Solving Micro-Consumption Models with Durables, Nondurables and Liquidity Constraints*

Christopher C. Farr
FieSta Solutions, LLC

María José Luengo-Prado[†]
Universidad Carlos III

June 1, 2001

Abstract

We propose two methods (Euler Equation Iteration and a Finite-State Approximation) for solving nonlinear rational expectations models of consumption behavior with durables, nondurables and a collateralized liquidity constraint which requires agents to make a down payment on durable purchases.

KEYWORDS: Buffer-stock, Consumption, Durable Goods, Computational Economics.

JEL CLASSIFICATIONS: E21, C36, C61.

---

[*] We thank John Driscoll and Adam Szpiro in the Applied Mathematics Department at Brown University. Luengo-Prado acknowledges financial support from the Bank of Spain.

[†] Corresponding author. E-mail: mjluengo@eco.uc3m.es. Universidad Carlos III, Departamento de Economía, C. Madrid 126, 28903 Getafe (Madrid), Spain. Phone: +34 91 6245738

1 Introduction

The buffer-stock model of savings, pioneered by Deaton (1991) and Carroll (1997), has become the standard framework for thinking about the way a typical consumer behaves. The model incorporates precautionary motives for savings and borrowing restrictions to the classical idea that consumption is determined by permanent income. The buffer-stock model can explain several empirical puzzles found in household data, including the excess sensitivity and the excess smoothness of consumption.[1] However, at the aggregate level, the implications of the buffer stock model are less satisfactory. For example, Ludvigson & Michaelides (1998) show that an aggregate version of the buffer stock model where agents observe each component of their income does not generate robust excesses. Many suggest that the absence of durable goods in the standard specification of the buffer stock model may be to blame for these results. However, incorporating durable goods into this framework is not straightforward, overall computationally.

This paper presents the necessary derivations and numerical procedures for solving a buffer-stock model of savings generalized to include nondurable goods and durable goods. Since partially collateralized loans are common for the financing of durables, we specify a collateralized liquidity constraint in the model. In particular, we assume that after satisfying a down payment requirement, consumers can finance the remaining fraction of durable purchases. Our model also allows for non-convex adjustment costs in the durable market *à la* Grossman & Laroque (1990). Several authors have used models with similar specifications to study diverse empirical questions but to our knowledge the model has not yet been solved.[2]

It is well known that under general forms of income uncertainty the standard buffer-stock model does not have an analytical solution and authors have utilized numerical methods to approximate answers. *Euler Equation Iteration* has been the technique of choice over the last decade.[3] This recursive method finds a functional fixed point to the intertemporal first-order condition of the optimization problem and it is widely used since it finds optimal policies quickly and precisely. Moreover, it is also relatively flexible in accommodating different forms of liquidity constraints.[4] We show that the Euler Equation Iteration method

---

[1] The excess sensitivity of consumption refers to the fact that changes in consumption are significantly correlated with lagged or predictable income, and the excess smoothness to the fact that consumption growth is much smoother than income growth. Both facts are in discrepancy with the predictions of the PIH.

[2] See (Brugiavini & Weber (1994), Alessie, Devereux & Weber (1997), Chah, Ramey & Starr (1995)).

[3] See Deaton & Laroque (1992).

[4] Fixed borrowing limits (exogenous, Deaton (1991), and endogenous, Carroll (1997)), and limits that

can be further extended to solve this generalized model with durables but only when there are no adjustment costs.

When there are adjustment costs, the solution to the agent's problem will take the form of an (S, s) rule. The value function will have a kink at s and Euler equation iteration, a method based on taking derivatives, cannot be used. Since discrete approximations handle kinks well, we show that a discrete dynamic programming technique can be used to solve the model. We call this technique *Finite-State Approximation.*

2 The Model

The consumer's problem is to maximize the present discounted value of expected utility from consumption of a nondurable good, C_t, and a durable good, K_t.[5] We will assume that time, t, is discrete and agents face an infinite horizon. $\beta < 1$ is their discount rate. The instantaneous utility function is assumed to be separable in both goods and it is of the CRRA type.[6] The maximization problem can be written as:

$$\max_{\{C_t, K_t\}} V = \mathrm{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\rho}}{1-\rho} + \frac{K_t^{1-\rho}}{1-\rho} \right) \right\}, \tag{1}$$

$$\text{s.t. } A_t = RA_{t-1} + Y_t - (K_t - \psi K_{t-1}) - d\phi\psi K_{t-1} - C_t \ , \tag{2}$$

$$A_t + K_t \geq \theta K_t \ , \ \theta \in [0, 1]. \tag{3}$$

Equation (2) is the budget constraint between two successive periods. A_t is a riskless financial asset and R is the interest factor paid on it. Y_t is labor income, which is assumed to be exogenous to the agent and a stationary random variable with range $[Y_L, Y_H]$, $Y_L \geq 0$.[7] $(K_t - \psi K_{t-1})$ is net investment on the durable and ψ is the depreciation factor. $d\phi\psi K_{t-1}$ is an adjustment cost paid when changing the durable stock and it is assumed to be proportional to the beginning-of-period value of the durable. d is an indicator variable which takes on a value of zero when there is no investment and a value of one when investment

---

vary stochastically with income (Ludvigson (1999)).

[5] We make the simplifying assumption that the agent's service flows from the durable are proportional to the durable stock held and set the constant of proportionality equal to one. This simplification has been used by many others. See for example Mankiw (1982) or Chah et al. (1995).

[6] We follow Bernanke (1984) who studied the joint behavior of the consumption of durable and nondurable goods and found that separability was a good approximation. The techniques explained here can be easily modified to deal with non-separable utility functions. We assume that prices are as follow: $P_t^C / P_t^K = 1, \forall t$.

[7] The solution techniques can be easily modified to accommodate nonstationary and serially correlated processes.

is non-zero. Equation (3) is the liquidity constraint. An individual's borrowing limit is a function the durable stock since this can act as collateral for credit purchases. θ can be seen as a down payment requirement.[8]

3 Solving the Model with No Adjustment Costs: Euler Equation Iteration

When there are no adjustment costs ($\phi = 0$), Euler Equation Iteration can be used to solve the problem. It will be convenient to reformulate the model as follows. Define cash-on-hand, X_t, as the sum of beginning-of-period financial wealth, durable wealth and current labor income: $X_t \equiv RA_{t-1} + \psi K_{t-1} + Y_t$. The budget constraint becomes $A_t = X_t - C_t - K_t$ and the liquidity constraint $C_t + \theta K_t \leq X_t$. Combining the budget constraint and the liquidity constraint, we can write an equation for the evolution of cash-on-hand $X_{t+1} = R(X_t - C_t) + (\psi - R)K_t + Y_{t+1}$.

Our objective is to find the optimal durable and nondurable consumption rules as a function of the only state variable X_t. The Bellman equation and the first-order conditions are:

$$\begin{aligned} V_t(X_t) &= \max_{\{C_t, K_t\}} U(C_t, K_t) + \beta \mathrm{E}_t \left\{ V_{t+1}[R(X_t - C_t) + (\psi - R)K_t + Y_{t+1}] \right\} \\ &\quad + \lambda_t \left(X_t - C_t - \theta K_t \right) . \end{aligned} \tag{4}$$

$$U_C^t = \beta R \, \mathrm{E}_t[U_C^{t+1}] + \lambda_t \; , \tag{5}$$

$$U_K^t = \beta(R - \psi) \, \mathrm{E}_t[U_C^{t+1}] + \theta \lambda_t \; , \tag{6}$$

$$\lambda_t(X_t - C_t - \theta K_t) = 0 \; . \tag{7}$$

Note the difference in the discount factor in the two Euler equations: βR versus $\beta(R - \psi)$ since the durable holds a proportion ψ of its value until the next period. Also, λ_t is multiplied by θ in equation (6) to reflect the fact that only a down payment is required as payment for the durable in period t.

Combining the first order conditions, we obtain an equation of the intratemporal relationship between C_t and K_t:

---

[8] Note that $\theta = 0$ would imply that durable purchases do not affect current liquidity

$$U_K^t = \frac{R-\psi}{R}\, U_C^t + \left(\theta - \frac{R-\psi}{R}\right)\, \lambda_t. \tag{8}$$

We can see that the liquidity constraint imposes distortions on the allocation of consumption across time and across goods. Also, note that when the liquidity constraint is not binding, $\lambda_t = 0$, and for our particular utility function

$$\frac{C_t}{K_t} = \left(\frac{R-\psi}{R}\right)^{\frac{1}{\rho}} \equiv \Omega\ .$$

This is the optimal relationship between C_t and K_t that accounts for durability.[9] $(R-\psi)/R$ is the user cost of the durable. This cost represents the single-period cost, or rental equivalent of a unit of the durable. Note that when $\theta = (R-\psi)/R$, the same relationship between C_t and K_t holds for a constrained agent and the liquidity constraint does not produce distortions across goods.

3.1 Numerical Implementation

When the agent is not liquidity constrained, $\lambda_t = 0$, and from equation (8) we know that $C_t/K_t = \Omega$. Equation (5) can be rewritten as

$$-C_t^{-\rho} + \beta R \mathrm{E}_t \left\{ \left(C_{t+1} \left[R(X_t - C_t) + \left(\frac{\psi - R}{\Omega}\right) C_t + Y_{t+1} \right] \right)^{-\rho} \right\} = 0\ . \tag{5$'$}$$

When the agent is constrained, $\lambda_t > 0$. Moreover, the liquidity constraint is binding: $X_t = C_t + \theta K_t$. Solving for λ_t in (5), plugging it into (6) and using the liquidity constraint, we can write:

$$\begin{aligned} &-\theta\, C_t^{-\rho} + \left(\frac{X_t - C_t}{\theta}\right)^{-\rho} \\ &+\beta\, [\psi - R(1-\theta)]\, \mathrm{E}_t \left\{ \left(C_{t+1} \left[[\psi - R(1-\theta)] \frac{X_t - C_t}{\theta} + Y_{t+1} \right] \right)^{-\rho} \right\} = 0\ . \end{aligned} \tag{6$'$}$$

With the two equations above, we are ready to find the optimal rule for nondurable consumption as a function of the unique state variable, cash-on-hand, X, which we denote

---

[9] If ψ was 0, i.e. the durable would depreciate completely after one period, $U_C^t = U_K^t$. The agent would choose to consume the same amounts of both goods ($C_t = K_t$). As long as $\psi > 0$, $C_t < K_t$.

$C(X)$.[10] The optimal policy function for the durable, $K(X)$, can be backed out by using the intratemporal relationship between the two goods.

$$K(X) = \begin{cases} \theta^{-1}[X - C(X)], & X \leq X^*, \\ \\ \Omega^{-1} C(X), & X \geq X^*, \end{cases}$$

Once we have written the Euler equations this way, applying the Euler equation iteration method is fairly simple. The technique requires assuming a finite horizon, T, and recursively solving backwards from the last period of life. To apply the method successfully, we need to (i) evaluate the expectation, (ii) select an appropriate terminal condition and (iii) find a criterion to check if the agent is liquidity constrained to decide which of the two equation above is to be used.

In order to evaluate the expectation, we replace the continuous income processes, Y_t, by a discrete approximations as suggested by Tauchen (1986). With regards to the terminal condition, we will assume that $C_T(X) = X$. In period $T-1$, for a given value of X, we can numerically compute the value C_{T-1} that satisfies the appropriate Euler equation: the first one if the agent is not constrained and the second one if he is. We do so for a grid of values of X and numerically approximate the optimal consumption rule $C_{T-1}(X)$ through interpolation between the points of the X grid. Once we have $C_{T-1}(X)$, the same grid of X values is used to compute $C_{T-2}(X)$. With $C_{T-2}(X)$, $C_{T-3}(X)$ is computed, and so on. Under appropriate conditions, as we iterate backwards, the policy function will converge to the solution to the infinite horizon problem.[11] Note that there is an easy way to check if the agent is liquidity constrained for a given value of X. At each time iteration, find X_t^*, the exact value of cash-on-hand for which the liquidity constraint just binds, which is $C_t(X_t^*) = \Omega(\Omega+\theta)^{-1}X_t^*$, and solve equation $(5')$ for X_t^*. For all $X_i \leq X_t^*$ the agent is constrained in this iteration, and *vice versa*.

---

[10] When $\theta = 0$, we cannot use $(6')$ as it is. We should rewrite $(6')$ as

$$K_t^{-\rho} + \beta(\psi - R)\,\mathrm{E}_t\left\{C_{t+1}\left[(\psi - R)K_t + Y_{t+1}\right]^{-\rho}\right\} = 0\,. \qquad (6'')$$

[11] In all our examples, the "impatience" condition common to buffer-stock models plus $\psi < R$ ensure convergence.

4 Solving the Model with Adjustment Costs: Finite-State Approximation

Now consider the model with adjustment costs ($\phi \neq 0$). In this instance, the solution to the agent's problem will take the form of an (S, s) rule, with the corresponding kink in the value function. Any method based on taking derivatives, such as Euler Equation Iteration, will have trouble dealing with points in the value function that are not differentiable. Therefore, it is better to work with the value function directly. The Finite-State Approximation method that we propose consists of solving a discrete version of the problem above by using infinite-horizon dynamic programing techniques. In particular, we use value function iteration in combination with a modified policy function acceleration technique.[12] Appendix A provides a complete description of the method.

In order to apply this technique successfully, we need to reformulate the agent's problem once more. In this case, we must work with two state variables. We choose Q_t, voluntary equity, defined as $Q_t \equiv A_t + (1-\theta)K_t$, and the durable stock, K_t.[13] The liquidity constraint can be written as $Q_t \geq 0$, $\forall t$, and the budget constraint becomes:

$$Q_t = RQ_{t-1} + [\psi(1-d\phi) - R(1-\theta)]K_{t-1} + Y_t - \theta K_t - C_t \; . \tag{9}$$

The Bellman equation of the model is given by:

$$\begin{aligned} V(Q_{t-1}, K_{t-1}) \quad = \quad & \beta\, \mathrm{E}_{t-1}\left\{ \max_{\{Q_t, K_t;\, Q_t \geq 0\}} U\left[\left\{ RQ_{t-1} + [\psi(1-d\phi) - R(1-\theta)]K_{t-1} \right\}\right.\right. \\ & \left.\left. -\theta K_t + Y_t - Q_t, K_t \right] + V(Q_t, K_t) \right\} , \end{aligned}$$

where we have substituted out for nondurable consumption into the instantaneous utility function from equation (9).[14] Note that control variables and state variables coincide in this formulation of the problem. Our goal is to find the function $V^*(Q, K)$, and the policy functions $Q^*(Q, K, Y)$ and $K^*(Q, K, Y)$ which satisfy the Bellman equation above.

---

[12] See Judd (1997) for a general description of these techniques.

[13] $(A_t,\ K_t)$ seems a more natural choice. However, this pair proves very inconvenient. The problem is that with $K_t > 0$ (from preferences) and $A_t \geq -(1-\theta)K_t$, the feasible region in the (A_t, K_t) space is not rectangular.

[14] It is easy to show that the necessary first-order conditions are exactly the same with this new specification of the problem.

4.1 Numerical Implementation

The first step is to discretize the problem. We construct a N_Y element grid for Y as in the previous section. We need a N_Q element grid for Q and a N_K element grid for K. To implement the liquidity constraint (i.e. $Q_t \geq 0$), the lower bound on the Q-grid is set to zero. We set the lower bound of the K-grid to 0 as well. The upper bounds of the grids must be selected so that the actual solution to the agent's problem lies 'inside' of the choice set defined by the grids.[15]

The adjustment cost poses an added difficulty. For each grid point on the K-grid, the agent can invest nothing and not suffer the adjustment cost, implying that for each durable grid point, K_j, there is another grid point, K_i, which satisfies $K_i - \psi K_j = 0$. When $\psi = 0$, any grid can satisfy this condition. However, when ψ is non-zero, it is difficult to construct a grid with the above property. Our solution is the following. First, we locate the value of an agent's durable stock, ψK_j, on the K-grid. The usual case is that this value falls between two grid points, let's say between grid point K_i and K_{i+1}. The agent is allowed to "move" to either of these grid points without paying the adjustment cost. Thus, the indicator variable, d, for the numerical implementation is given by

$$d = \begin{cases} 0, & \text{when } K_t - \psi K_{t-1} \leq \kappa \\ 1, & \text{when } K_t - \psi K_{t-1} > \kappa, \end{cases} \tag{10}$$

where κ is the step of the K-grid.

Note, that the precision of our solution increases as κ falls. This "work around" solution may have some economic significance. It may be possible for the agent to make small changes to his durable stock, such as repairs, which do not require significant adjustment costs. If this is the case, the numerical formulation described here would be most appropriate.

The grid density, or the step between grid points, also needs to be determined. Precision of the approximation increases as the step between grid points falls. The cost of precision is the longer computational time as there are more grid points. Table 1 illustrates three different step sizes and the associated time to complete one iteration. In this table $\phi = 0$ is used to compare both methods. Note that the time increases exponentially as the step size is reduced in half.[16] The statistics obtained from simulations do not differ by much for the three cases. Precision, measured as how well the optimal policy rules satisfy the

---

[15] We calculate the joint ergodic distribution for Q and K implied by the stochastic income process and the optimal policy functions and verify that the bounds are appropriate.

[16] Each time the step size (for both dimensions) is reduced by half, calculation time increases by a factor of 2^4.

intratemporal first order condition when the agent is not liquidity constrained, improves as the grid step decreases. Euler Equation Iteration produces a more precise answer.

5 Concluding Remarks

We show how to solve a generalized buffer stock model of savings that includes durables, nondurables and a collateralized liquidity constraint with two different methods. Euler Equation Iteration is a fast method that produces precise solutions to intertemporal consumption models with nonlinear Euler equations. However, not every problem can be mapped into the Euler Equation Iteration framework. For example, in the adjustment cost model of durable consumption that we consider, where the policy function is multidimensional and exhibits threshold properties, Euler Equation Iteration does not work. We illustrate how a Finite-State approximation method, slower but more robust may allow us to find solutions to this kind of problems with relatively good precision.

Table 1: comparing techniques when there are not adjustment costs.

		FSA		EEI
	coarse	medium	fine	
Q-grid step / μ_Y	0.04	0.02	0.01	
K-grid step / μ_Y	0.08	0.04	0.02	
No. of elements in state set	625	2,500	10,000	
Average time per iteration (min.)	0.02	0.33	5.15	0.001
Iterations until convergence	12	13	13	41
Statistics from Simulations				
Average C	76.50	76.47	76.47	76.47
	(6.02)	(5.98)	(5.98)	(6.11)
Average K	236.11	236.30	236.26	236.28
	(22.45)	(22.39)	(22.31)	(22.25)
First-Order Condition Check				
Average $\lvert\Omega - C^*/K^*\rvert$ when $Q_i \neq 0$ (in simulations)	4.7E-03	2.3E-03	1.2E-03	1.47E-17

Notes: FSA: Finite-State Approximation. EEI: Euler Equation Iteration. $R = 1.02$, $\psi = 0.9145$, $\phi = 0$, $\theta = 0.2$, $\beta = 0.9524$, $\rho = 2$, $Y \sim N[100, 20]$, $\Omega = ((R - \psi)/R)^{1/\rho} = 0.3216$. $\mu_y = 100$ is mean income. Simulation results are averages of 1,000 samples of 10,000 periods each. $Q_i \in [0, 100]$, $K_j \in [150, 350]$. All calculations were performed in a Pentium II-300 with 64MB of Ram. All programs are in C++.

Appendix A. The Finite-State Approximation Algorithm.

Objective: To find the optimal policy function, $U^* = (Q^*(Q,K,Y), K^*(Q,K,Y))$, and the value function $V^*(Q,K)$ that solve the Bellman equation.

Initialization: Discretize Q, K, and Y. Choose an initial value function, V^0. Set $V^\ell = V^0$. Decide on a stopping criterion, $\varepsilon > 0$, $\varepsilon \to 0$.

Step 1: Find policy iterate, $U^{\ell+1}$, (GRID SEARCH).

- Employ the mapping $\mathcal{U}$ to V^ℓ to find $U^{\ell+1}$ (i.e. $U^{\ell+1} = \mathcal{U}V^\ell$). For each (Q_i, K_j) and Y_m, the mapping $\mathcal{U}$ is defined as

$$\begin{aligned} U^{\ell+1}_{i,j,m} \quad &= \max_{\{Q^+,K^+;\,Q^+\geq 0\}} U\Big[RQ_i + [\psi(1-d\phi) - R(1-\theta)]K_j - \theta K^+ + Y_m - Q^+, K^+\Big] \\ &\quad + \; V^\ell(Q^+,K^+) \;\equiv\; \mathcal{U}V^\ell_{i,j} \end{aligned}$$

Step 2: Find value iterate, $V^{\ell+1}$ (ACCELERATION TECHNIQUE).

- Let $W^1 = V^\ell$. For each (Q_i, K_j),

$$\begin{aligned} W^{k+1}(Q_i,K_j) \quad &= \quad \beta(1/N_Y)\Big\{\sum_{m=1}^{N_Y} U\Big[RQ_i + [\psi(1-d\phi) - R(1-\theta)]K_j - \theta U^{\ell+1}_K \\ &\quad + Y_m - U^{\ell+1}_Q, U^{\ell+1}_K\Big] + W^k[U^{\ell+1}_Q, U^{\ell+1}_K]\Big\}\,. \end{aligned}$$

- Do for $k = 1, \ldots, S$. Let $V^{\ell+1} = W^S$.

Step 3: Calculate the error between successive value functions.

If $\|V^{\ell+1} - V^\ell\| < \varepsilon$, stop. $U^* = U^{\ell+1}$. Otherwise go to *Step 4.*

Step 4: Set $V^\ell = V^{\ell+1}$ and go to *Step 1.*

References

Alessie, R., Devereux, M. P. & Weber, G. (1997), "Intertemporal Consumption, Durables and Liquidity Constraints: A Cohort Analysis", *European Economic Review* **41**, 37–59.

Bernanke, B. S. (1984), "Permanent Income, Liquidity, and Expenditure on Automobiles: Evidence From Panel Data", *Quarterly Journal of Economics* **99**(3), 587–614.

Brugiavini, A. & Weber, G. (1994), Durables and Non-Durables Consumption: Evidence From Italian Household Data, *in* "Saving and the Accumulation of Wealth", Cambridge University Press, chapter 10, pp. 305–329.

Carroll, C. D. (1997), "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis", *Quarterly Journal of Economics* **112**(1), 1–55.

Chah, E. Y., Ramey, V. A. & Starr, R. M. (1995), "Liquidity Constraints and Intertemporal Consumer Optimization: Theory and Evidence from Durable Goods", *Journal of Money, Credit, and Banking* **27**(1), 272–287.

Deaton, A. (1991), "Saving and Liquidity Constraints", *Econometrica* **59**(5), 1121–1248.

Deaton, A. & Laroque, G. (1992), "On the Behaviour of Commodity Prices", *Review of Economic Studies* **59**, 1—23.

Grossman, S. J. & Laroque, G. (1990), "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods", *Econometrica* **58**(1), 25–51.

Judd, K. L. (1997), *Numerical Methods in Economics*, MIT Press.

Ludvigson, S. (1999), "Consumption and Credit: A Model of Time-Varying Liquidity Constraints", *The Review of Economics and Statistics* **81**(3), 434–447.

Ludvigson, S. & Michaelides, A. (1998), "Can We Explain the Consumption Excesses of the 1980s? Aggregate Implications of Buffer Stock Saving Behavior". Manuscript.

Mankiw, N. G. (1982), "Hall's Consumption Hypothesis and Durable Goods", *Journal of Monetary Economics* **10**, 417–425.

Tauchen, G. (1986), "Finite State Markov Chain Approximations to Univariate and Vector Autoregressions", *Economic Letters* **20**, 177–181.